
SQUARE-ROOT EQUATIONS

An equation such as $\sqrt{x-4} - 5 = 0$, with the variable inside the square-root sign, is called a **square-root equation** (or, in general, a *radical equation*). Just as subtraction in an equation is removed by addition, a square root is removed by squaring.



Here are the major steps to solve a general radical equation:

1. Isolate the radical (if necessary).
2. Raise each side of the equation to an appropriate power (squaring in this chapter).
3. Solve the resulting equation for potential solutions.
4. Check every potential solution in the original equation.

□ EXAMPLES

EXAMPLE 1: Solve for z : $\sqrt{z} = 7$

Solution: The radical is already isolated, so on to step 2. Since the radical is a square root, we will square each side of the equation:

The key to solving a square-root equation is the fact that, assuming $x \geq 0$,

$$(\sqrt{x})^2 = x$$

$$\sqrt{z} = 7 \quad \text{(the original equation)}$$

$$\Rightarrow (\sqrt{z})^2 = 7^2 \quad \text{(square each side of the equation)}$$

$$\Rightarrow \boxed{z = 49} \quad \text{(squaring undoes square rooting)}$$

Put 49 back into the original equation, and you'll see that we have a valid solution.

EXAMPLE 2: Solve for x : $\sqrt{x-4} - 5 = 0$

Solution: Our plan is to isolate and square.

Start with the original equation: $\sqrt{x-4} - 5 = 0$

First, isolate the radical: $\sqrt{x-4} = 5$

Second, to remove the square root sign,

square each side of the equation: $(\sqrt{x-4})^2 = 5^2$

Simplify each side of the equation: $x - 4 = 25$

Now solve for x : $\underline{x = 29}$

Now it's time to check our solution. Let's put our candidate, 29, into the original equation and perform the arithmetic separately on each side of the equation:

$\sqrt{29-4} - 5$	$\underline{0}$
$\sqrt{25} - 5$	
$5 - 5$	
$\underline{0}$	✓

So 29 checks out; therefore, the solution is $x = 29$

EXAMPLE 3: Solve for a : $\sqrt{a^2 + 15} + 10 = 2$

Solution: We isolate the radical, square each side, solve for a , and then CHECK the solutions.

$$\begin{aligned}
 & \sqrt{a^2 + 15} + 10 = 2 && \text{(the original equation)} \\
 \Rightarrow & \sqrt{a^2 + 15} = -8 && \text{(isolate the radical)} \\
 \Rightarrow & \left(\sqrt{a^2 + 15}\right)^2 = (-8)^2 && \text{(square each side)} \\
 \Rightarrow & a^2 + 15 = 64 && \text{(simplify each side)} \\
 \Rightarrow & a^2 - 49 = 0 && \text{(subtract 64 from each side)} \\
 \Rightarrow & (a + 7)(a - 7) = 0 && \text{(factor)} \\
 \Rightarrow & a + 7 = 0 \text{ or } a - 7 = 0 && \text{(set each factor to 0)} \\
 \Rightarrow & \underline{a = -7 \text{ or } a = 7} && \text{(solve for } a\text{)}
 \end{aligned}$$

These are two possible solutions, 7 and -7 . Let's check out the 7:

$\sqrt{7^2 + 15} + 10$	$\underline{2}$
$\sqrt{49 + 16} + 10$	
$\sqrt{64} + 10$	
$8 + 10$	
$\underline{18}$	✗

The two sides of the equation do not balance; therefore, 7 is not a solution of the radical equation. Using -7 for a will result in the same calculations, indicating that -7 is also not a solution. So, what do we have here? We solved an equation and got two potential solutions, 7 and -7 , but neither of them panned out (they are called *extraneous solutions*). That's it — it's over — nothing works.

No Solution

EXAMPLE 4: Solve for x : $\sqrt{11-x} + 5 = x$

Solution: To isolate the radical, we need to subtract 5 from each side of the equation:

$$\sqrt{11-x} = x-5$$

Since we now have a square root to undo, we will square each side of the equation:

$$(\sqrt{11-x})^2 = (x-5)^2 \quad \text{Notice the parentheses around the } x-5.$$

Now we simplify each side of the equation:

$$11-x = x^2 - 10x + 25 \quad \text{Careful when squaring } x-5.$$

Recognizing this equation as quadratic, we'll put it into standard form by adding x to each side and subtracting 11 from each side:

$$0 = x^2 - 9x + 14$$

Factoring produces

$$0 = (x-7)(x-2)$$

You're welcome to switch the sides and write

$$x^2 - 9x + 14 = 0$$

Setting each factor to 0 and solving gives us two candidates for a solution:

$$\underline{x = 7} \quad \text{or} \quad \underline{x = 2}$$

We now check each potential solution in the original equation:

$$\underline{x = 7}$$

$$\begin{array}{l|l} \sqrt{11-\color{blue}{7}} + 5 & \underline{7} \\ \sqrt{4} + 5 & \\ 2 + 5 & \\ \underline{7} & \checkmark \end{array}$$

$$\underline{x = 2}$$

$$\begin{array}{l|l} \sqrt{11-\color{blue}{2}} + 5 & \underline{2} \\ \sqrt{9} + 5 & \\ 3 + 5 & \\ \underline{8} & \times \end{array}$$

What does all this mean? Easy — the **7** works while the **2** doesn't. So, even though we got two solutions when we solved the radical equation, only one actually worked out.

The 2 we obtained but failed to work in the original equation is called an *extraneous solution*. (By the way, just because one solution failed to work does not mean we made any mistakes when solving the problem.) Therefore, the final solution of the equation is

$$x = 7$$

Homework

Solve and check each equation:

1. $\sqrt{x} = 10$

2. $\sqrt{y} = 5$

3. $\sqrt{t} = 0$

4. $\sqrt{a} = -3$

5. $\sqrt{x+1} = 3$

6. $\sqrt{c-5} = 10$

7. $\sqrt{x+5} - 2 = 3$

8. $\sqrt{2y-1} + 3 = 6$

9. $\sqrt{8-n} - 4 = 0$

10. $\sqrt{7z+100} + 10 = 20$

11. $\sqrt{5x-1} = \sqrt{7x-25}$

12. $\sqrt{x} + 2 = x$

13. $\sqrt{n+1} + 3 = n - 8$

14. $\sqrt{2x-11} + x = 13$

□ **TO ∞ AND BEYOND**

A. Solve for x : $\sqrt[5]{7x-9} = 2$

B. Solve for x : $\sqrt{x+40} = 4 + \sqrt{x}$

C. Solve for y : $\sqrt[3]{(y-2)^2} = 4$

Solutions

1. $x = 100$ 2. $y = 25$ 3. $t = 0$ 4. No solution
5. $x = 8$ 6. $c = 105$ 7. $x = 20$ 8. $y = 5$
9. $n = -8$ 10. $z = 0$
11. Square each side: $5x - 1 = 7x - 25 \Rightarrow x = 12$; check it in the original equation, doing just simple arithmetic on each side separately. You should find that $x = 12$ works.
12. Isolate the radical and square each side, leading to the equation $x^2 - 5x + 4 = 0$, whose solutions are 4 and 1. Only the 4 works; the value 1 is called an *extraneous* solution. The final answer is $x = 4$.
13. Isolate the radical and square both sides — when simplified, you'll have the quadratic equation $n^2 - 23n + 120 = 0$. Only $n = 15$ works.
14. The two candidates for a solution are 10 and 18. The 18 is extraneous and the final answer is $x = 10$.

“The only person who is educated is the one who has learned how to learn . . . and change.” - Carl Rogers, Psychologist